Theory for the Estimation of Structural Vibration Parameters from Incomplete Data

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Introduction

IN system identification and modal analysis, the investigator is often faced with the problem of having to construct a model of finite order to represent a continuous vibrating structure. In other words, he is charged with obtaining a reduced-order model which, by definition, must be limited in its representation of the continuous system to a finite range of frequencies. The quality of the reduced-order model will be contingent upon the completeness of the measured data: a common cause of incompleteness occurs when the number of modes excited in the test is fewer than the number of measurement stations (i.e., less than the order of the identified model).

When it is required to construct a structural model (in terms of mass, stiffness, and damping parameters) from measured data, it is vital that such a model should in some sense be physically meaningful. The problems associated with the construction of reduced-order structural models from incomplete data have been highlighted by Berman and Flannelly[1] and Berman.[2] It is obvious that a reduced-order model cannot possibly reproduce the behavior of a continuous structure across an infinite range of frequencies. In the real structure the mass is distributed, whereas in the model the mass is discretized. In the finite-element method, consistent mass and stiffness matrices may be formulated by the minimization of an energy functional and, for this reason, such matrices may be considered to possess physical meaning. In reality, finite-element models do not exactly represent the dynamics of vibrating structures because they are incapable of dealing with boundary conditions such as imperfect hinges or flexibility in welded joints. It is then necessary to modify the finite-element model slightly in order that it should replicate the observed dynamics of the vibrating structure in a required range of frequencies.

Methods have been developed for the estimation of structural parameters based on measured modal data. Chen and Garba[3] presented a least-squares technique that may be applied iteratively together with matrix perturbation (for computation of the Jacobian matrix and reanalysis of the eigenvalues and eigenvectors) until the measured and computed eigendata are sufficiently in agreement. However, modal methods are known to have difficulty in treating closely spaced modes and can only deal with out-of-range modes by the addition of residual flexibilities.

In this Note, a method is presented that does not depend on a modal decomposition but uses measured frequency responses to construct a structural model that varies minimally from an initial finite-element representation. A model constructed by this method is thus physically meaningful in the sense that a finite-element model is meaningful. A simple problem is contrived to illustrate an extreme case of incomplete data, which is solved using a continuous-frequency- domain, least-squares filter.[4] Using this approach, the Jacobian matrix is a constant matrix. Other problems associated with measure-

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ment noise are not discussed here since the filtering method has already been shown to perform well when using experimental data from a portal frame rig.[5]

Theory

Consider an m-degree-of-freedom vibrating system that may be described in a finite range of frequencies by the following model

$$q(\omega) = B(\omega)[z(\omega) - \xi]$$ (1)

where $q(\omega)$ is an n-vector of input forces, $z(\omega)$ an n-vector of measured displacement responses, ξ an n-vector of measurement noise, $B^{-1}(\omega)$ an $n \times n$ matrix of frequency response functions, and $n < m$.

In the case of viscous damping,

$$B(\omega) = -\omega^2 M + j\omega C + K$$ (2)

where M, C, and K are $n \times n$ matrices of mass, viscous damping, and stiffness, respectively.

Equation (1) may be arranged as follows:

$$H^H x = q - {}^H y$$ (3)

where x is an ℓ-vector of *unknown* mass, damping, and stiffness parameters and y is a $(k-\ell)$-vector of *known* mass, damping, and stiffness parameters. For a symmetric system, $k = 3(n^2 + n)/2$.

$$H = \frac{d(Bz)^H}{dx}$$ (4)

$$= \frac{d(Bz)^H}{dy}$$ (5)

where H is the Jacobian matrix. Clearly Eq. (3) allows for imperfect measurement of the vibration responses.

If $r_i = q_i - D_i^H y$, then a least-squares estimate $\hat{x}$ of x can be obtained by solving the following:

$$\sum_{i=1}^{N} H_i^H \hat{x} = \sum_{i=1}^{N} r_i$$ (6)

where frequency-domain data is sampled at N intervals.

When an initial (finite-element) model x_{fe} is available then subtracting

$$\sum_{i=1}^{N} H_i^H x_{fe}$$

from both sides of Eq. (6) and multiplying throughout by $H_i Q$

$$\sum_{i=1}^{N} H_i Q H_i^H (\hat{x} - x_{fe}) = \sum_{i=1}^{N} H_i Q (r_i - H_i^H x_{fe})$$ (7)

where Q is a positive definite weighting matrix.

If $e = (\hat{x} - x_{fe})$ represents a least-squares correcting term that must be added to the finite-element model, then

$$e = \left(\sum_{i=1}^{N} H_i Q H_i^H \right)^+ + \sum_{i=1}^{N} H_i Q (r_i - H_i^H x_{fe})$$ (8)

If rank $\Sigma_{i=1}^{N} H_i Q H_i^H) = \ell$, then $(\ )^+ = (\ )^{-1}$, and a unique solution is available for e. It is more likely that $\Sigma_{i=1}^{N} H_i Q H_i^H$ is rank deficient, in which case there will be a multiplicity of least-squares solutions and then the pseudoinverse should be computed using singular value decomposition, which results in the unique solution of smallest norm $\|e\|_2$. Thus the singular value decomposition forces the solution $\hat{x}$, which deviates least from the initial model x_{fe}. The reader is referred to Ref. 6, pp. 324–325, for a discussion of the application of singular value decomposition to degenerate least-squares problems.

The result [Eq. (8)] may be posed as a discrete, frequency domain filter. Consider the case of a single frequency step from $i=j-1$ to $i=j$; then the estimate $\hat{x}_j$ can be computed from $\hat{x}_{j-1}$ as follows:

$$\hat{x}_j - \hat{x}_{j-1} = \left(\sum_{i=1}^{N} H_i Q H_i^H \right) + H_j Q (r_j - H_j^H \hat{x}_{j-1}) \tag{9}$$

or

$$\Delta e_j = \left(\sum_{i=1}^{N} H_i Q H_i^H \right) + H_j Q [r_j - H_j^H (x_{\text{fe}} + e_{j-1})] \tag{10}$$

where

$$e_{j-1} = (\hat{x}_{j-1} - x_{\text{fe}}) \tag{11}$$

$$e_j = e_{j-1} + \Delta e_j \tag{12}$$

If the frequency interval $\Delta\Omega = \Omega_j - \Omega_{j-1}$, then

$$\frac{\Delta e_j}{\Delta\Omega} = \left(\sum_{i=1}^{N} H_i Q H_i^H \Delta\Omega \right)$$

$$+ H_j Q [r_j - H_j^H (x_{fe} + e_{j-1})] \tag{13}$$

and in the limit as $\Delta\Omega \rightarrow 0$,

$$\frac{de}{d\Omega} = (P^{-1})^+ HQ[r - H^H(x_{\text{fe}} + e)] \tag{14}$$

$$\frac{dP^{-1}}{d\Omega} = HQH^H \tag{15}$$

Equations (14) and (15) together describe a continuous, frequency-domain filter for the estimation of the correcting term e.

Simulated Experiment

A simulated experiment was conducted using a finite-element representation of a simply supported beam. The beam of length $\ell = 32$, mass per unit length $m = 0.1$, and rigidity $EI = 1.47 \times 10^6$ was modeled using eight beam elements. The finite-element model contained 14 degrees-of-freedom (DOF), but it was required to set up a 3-degree-of-freedom model as shown in Fig. 1. Unit white noise excitation was applied in the transverse direction at the center of the beam such that the even modes were not exicited. A 3-DOF finite-element model was set up using Guyan reduction, and the lower triangles of the mass and stiffness matrices were entered into x_{fe}. The correcting term e was then computed using displacement data that were generated by the 14-DOF model but recorded only at the three measurement stations of the reduced model. Finally the two finite-element models and the least-squres model $\hat{x}$ were compared when the applied load was removed from the center of the beam and reapplied at one-quarter span. In this latter case, the first three modes were all excited.

In the processing of the measured data, two small singular values were set to zero resulting in a reduction of the condition number from 10^{16} to around 200. A typical acceleration spectrum for the two finite-element models and the corrected 3-DOF model in the range 0–500 rad/s with loading at midspan is shown in Fig. 2a. The natural frequencies (which were obtained from the eigenvalues) of the same three models are given in Table 1, and reduced finite-element and corrected parameters are provided in Table 2. It should be noted that the corrected model retains the natural symmetry of the simply supported beam problem (i.e., $k_{11} = k_{33}$, $k_{21} = k_{32}$, $m_{11} = m_{33}$, and $m_{21} = m_{32}$). Figure 2b shows an acceleration spectrum obtained after the load had been moved to one-quarter span.

The corrected model is shown to be significantly more accurate than a Guyan-reduced model in predicting the first and
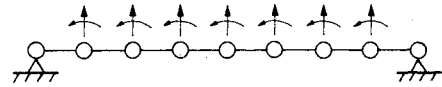

Fig. 1a   Finite-element models, 14-degree-of-freedom model.
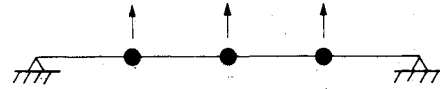

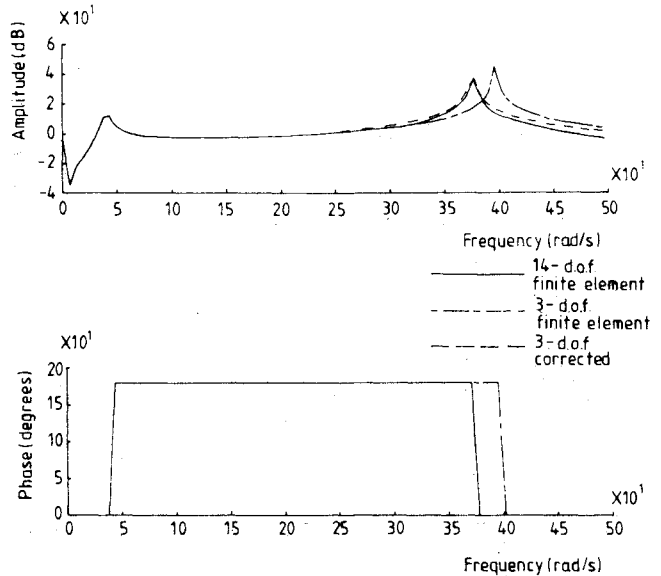Fig. 1b   Finite-element model, reduced 3-degree-of-freedom model.

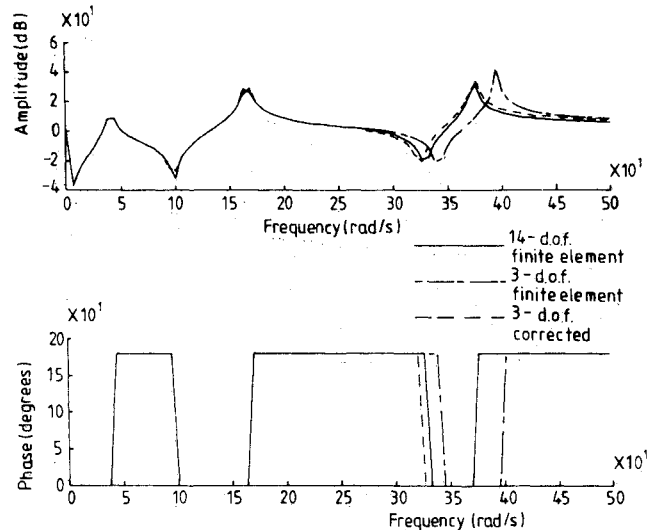Fig. 2a   Sample acceleration spectra, loading at midspan.

Fig. 2b   Sample acceleration spectra, loading at one-quarter-span.

Table 1   Computed natural frequencies

Finite element			Corrected	
14 DOF	3 DOF		3 DOF	
rad/s	rad/s	% error	rad/s	% error
41.74	41.75	0.03	41.74	0.0008
166.99	168.15	0.69	174.77	4.6600
376.16	397.62	5.71	376.15	− 0.0030

Table 2 Estimated parametrs

	Reduced finite-element model	Correcting term	Corrected model
	× 10⁵		
k_{11}	0.28351	−0.01343	0.27008
k_{21}	0.27119	0.02859	−0.24259
k_{22}	0.39445	−0.04031	0.35414
k_{31}	0.11094	−0.02686	0.08408
k_{32}	−0.27119	0.02859	−0.24259
k_{33}	0.28351	−0.01343	0.27008
	× 10⁻¹		
m_{11}	5.8509	0.01424	5.8651
m_{21}	0.47902	0.02570	0.50472
m_{22}	5.5981	0.04427	5.6423
m_{31}	−0.25279	0.02848	−0.22432
m_{32}	0.47902	0.02570	0.50472
m_{33}	5.8509	0.01424	5.8651

third eigenvalues. In the computation of the second eigenvalue, the reduced model is marginally better than the corrected model. Overall, the corrected model is superior to the reduced model in replicating the measured acceleration spectra. The estimated parameters deviate minimally from the parameters of the initial model and may be considered to be physically meaningful in the sense that a finite-element model is meaningful. As a final test of goodness, the model is able to predict accurately the response of the beam to loading conditions which differ from the loads applied in the test.

Acknowledgment

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Review of Composite Rotor Blade Modeling

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Introduction

HELICOPTER rotor blades are typically built-up, composite structures and made of materials that may be

anisotropic and nonhomogeneous. The initial impetus for the use of composites was the very significant improvement in fatigue life and damage tolerance of the blades and later the benefits afforded by the ability to incorporate more refined aerodynamic design into planform and airfoil section geometries. In recent years, the subject of aeroelastic tailoring has received attention (see Ref. 1). For advanced rotor blades, composite materials provide opportunities for structural simplicity of hingeless and bearingless designs and structural couplings to improve the aeroelastic stability and response of these configurations (see Ref. 2).

Most previously developed structural models have been limited to isotropic material properties. In modern rotor blades and flex beams, there may be coupling between extension, bending, torsion and shear deformation; warping effects may be much more significant. These complexities generally invalidate the Euler-Bernoulli beam assumptions that plane cross sections remain plane and perpendicular to the elastic axis. This paper is concerned with reviewing the modeling of such composite blades having arbitrary cross sections. A structural theory that is sufficiently general to treat such blades, with their varieties of cross sections, spanwise nonuniformities, and potentially large deflections, does not yet exist.

Research has yielded significant advances over the years in our ability to analyze beams with *simple* cross-sectional geometries such as circular tubes, channels, I-beams, rectangular boxes, cruciforms, etc. In particular, certain theoretical results have been systematically and thoroughly validated by correlation of experimental data with numerical simulation, even down to the prediction of strain at points.[3] However, successful analysis of these simpler structures does not necessarily imply that application of the same analyses to general, composite rotor blades will also be successful. The composite rotor-blade problem is of enormous complexity so that a "frontal assault" in which all the power of continuum mechanics and analysis is rigorously brought to bear may be impractical due to the computational effort that would be required. Certainly three-dimensional, finite-element methods such as the method in Ref. 4 could be employed, but this "brute force" technique is quite expensive, and the form of the results is not amendable to easy interpretation. Since rotor blades generally are much longer than their lateral dimensions, a one-dimensional model would seem feasible, a least from a computational point of view.

Previous Work

Although one-dimensional (i.e., beam) kinematics can be formulated in a rather elegant fashion,[5,6] constitutive laws in terms of known, three-dimensional elastic constants for small strains can only be approximations if the structure is to be treated as one-dimensional. Here the term "constitutive law" refers to the broad class of relations between generalized stress and strain which, for beam problems, entails knowledge of the shear center, various "warp" functions for both in-and out-of-plane cross-section deformation, and possibly as many as several dozen "stiffnesses" derived from various integrals over the cross section of the beam weighted by material moduli and various powers of cross-section coordinates and warp functions and their derivatives. Here and throughout the paper the term warp is used to denote not only in the usual sense of out-of-plane cross-sectional deformation, but also in the more general sense of in-plane, cross-sectional deformation.

In Refs. 6-8, a kinematical basis for beam theory is derived based on the concept of decomposition of the rotation tensor. The beam cross section is postulated to displace and rotate as a rigid body without explicit restriction on the magnitude of the motion. Further deformation of the cross section can be characterized by quantities whose magnitude is small in some sense. In particular, the total rotation at any point in the beam is represented as 1) a large global rotation of the reference triad, which is associated with a frame whose displacement